

# Coupled oscillators as models of quintom dark energy

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## Abstract

We investigate quintom cosmology in FRW universes using isomorphic models consisting of three coupled oscillators, one of which carries negative kinetic energy. In particular, we examine the cosmological paradigms of minimally-coupled massless quintom, of two conformally-coupled massive scalars and of conformally-coupled massive quintom, and we obtain their qualitative characteristics as well as their quantitative asymptotic behavior. For open or flat geometries, we find that, independently of the specific initial conditions, the universe is always led to an eternal expansion.

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# 1 Introduction

The type Ia supernova observations suggests that the universe is dominated by dark energy with negative pressure, which provides the dynamical mechanism for its accelerating expansion [1, 2, 3]. The strength of this acceleration is presently a matter of debate, mainly because it depends on the theoretical model implied when interpreting the data.

The most obvious theoretical candidate for dark energy is the cosmological constant  $\lambda$  (or vacuum energy) [4, 5] which has the equation of state  $w = -1$ . However, as it is well known, there are two difficulties arising from the cosmological constant scenario, namely the two famous cosmological constant problems — the “fine-tuning” and the “cosmic coincidence” one [6]. An alternative proposal is the concept of dynamical dark energy. Such a scenario is often realized by some scalar field mechanism and suggests that the energy form with negative pressure is provided by a scalar field evolving under a properly constructed potential. So far, a large class of scalar-field dark energy models have been studied, including quintessence [7], K-essence [8], tachyon [9], phantom [10], ghost condensate [11], quintom [12], and so forth. It should be noted that the usual viewpoint regards scalar-field dark energy models as an effective description of an underlying theory of dark energy. In addition, other proposals on dark energy include interacting dark energy models [13], braneworld models [14], Chaplygin gas models [15], holographic dark energy [16], bulk holographic dark energy [17] and many others. In this context, scalar fields, which may come in different forms and with a variety of possible self-interaction potentials, constitute the dominant (sole) form of matter in the field equations of the gravitational theory.

In the present work we consider scalar fields as the ones responsible for a dark energy universe behaving as quintom, that is obtained from an interplay of phantom and quintessence models. Neither of these two models alone can fulfill the transition from  $w > -1$  to  $w < -1$  and vice versa. Furthermore, although in k-essence [8] one can have both  $w \geq -1$  and  $w < -1$ , it has been lately shown in [18, 19] that the corresponding crossing is very unlikely to be realized during the evolution. However, one can show [12, 20] that considering the combination of quintessence and phantom in a qualitatively new model, the  $-1$ -transition can be fulfilled, as can be clearly seen in [12]. The quintom scenario of dark energy is designed to understand the nature of dark energy with  $w$  across  $-1$ . The quintom models of dark energy differ from the quintessence, phantom and k-essence and so on in the determination of the cosmological evolution and the fate of the universe.

Under the assumption of a quintom field with negative kinetic energy, the demand for a final dominance of phantom universe leads naturally to the consideration of a minimally-coupled quintom field, too. In the non-spatially flat universe, the problem of quintom stability has motivated the formulation of a toy-model [21] consisting of three coupled oscillators, one with negative and the others with positive-definite kinetic energy, with the first oscillator mimicking the gravitational field while the other two mimic the quintom (phantom and quintessence) field. In section 2 we summarize the equations for the coupled-oscillator models. In particular in subsection 2.1 we formulate the dynamics of the massless quintom scenario, while in 2.2 we construct an oscillator-model with positive energy which describes exactly a conformally-coupled quintom in rescaled variables and conformal time. In section 3 we investigate the characteristics of the isomorphic model of the massless quintom and in section 4 of that corresponding to the conformally-coupled quintom. Finally, in section 5 we summarize the obtained results and we discuss their physical implications.

## 2 Coupled oscillators as cosmological models

We know that toy-models, in which coupled oscillators are utilized, are present in recent cosmological studies. In this section we rewrite the field equations relating the quintom model and scalar field cosmology and we show that these equations can be reduced to the equations of a model with three coupled oscillators. We perform our investigation in the framework of a non-spatially-flat Friedmann-Robertson-Walker (FRW) universe which is

described by the line element

$$ds^2 = -dt^2 + a^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right). \quad (1)$$

## 2.1 Quintom cosmology

In order to describe the quintom field we use, as usual, two scalars:  $\phi$  and  $\sigma$ . For simplicity we assume that they constitute the only form of matter present. The energy density and pressure of this minimally-coupled quintom field in the metric (1) are respectively:

$$\rho = -\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\dot{\sigma}^2 + V(\phi, \sigma) \quad (2)$$

$$P = -\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\dot{\sigma}^2 - V(\phi, \sigma), \quad (3)$$

which correspond to the normal definitions for minimally-coupled scalars but with sign-inversion of the kinetic energy of  $\phi$ . In a homogeneous and isotropic universe  $\phi(t)$  and  $\sigma(t)$  depend only on the comoving time  $t$  and interact through the potential  $V(\phi, \sigma)$ .  $V(\phi, \sigma)$  should not be more negative than a norm because, due to the final dominance of phantom universe, the energy density  $\rho$  should be non-negative and the Hubble parameter  $H$  should be real. Note that a minimally-coupled scalar with positive kinetic energy density  $+\frac{1}{2}\dot{\phi}^2$  cannot exhibit  $P < -\rho$  in Einstein gravity [22].

For a quintom-universe described by metric (1), with the  $\phi$  and  $\sigma$  fields as material sources, the equations of motion are the following:

$$H^2 + \frac{k}{a^2} = \frac{\kappa}{6} [\dot{\sigma}^2 - \dot{\phi}^2 + 2V(\phi, \sigma)] \quad (4)$$

$$\dot{H} + H^2 = \frac{\kappa}{3} [-\dot{\sigma}^2 + \dot{\phi}^2 + V(\phi, \sigma)] \quad (5)$$

$$\ddot{\phi} + 3H\dot{\phi} - \frac{\partial V}{\partial \phi} = 0 \quad (6)$$

$$\ddot{\sigma} + 3H\dot{\sigma} + \frac{\partial V}{\partial \sigma} = 0, \quad (7)$$

where  $\kappa = 8\pi G$  and dots denote differentiation with respect to the comoving time  $t$ . It is remarkable that only three of the equations (4)-(7) are independent. Indeed, when  $\dot{\phi} \neq 0$  or  $\dot{\sigma} \neq 0$  one can derive one of the Klein-Gordon equations (6),(7) using the other and either (4),(5) or the conservation equation  $\dot{\rho} + 3H(P + \rho) = 0$  satisfied by the quintom. Furthermore, a significant feature of equations (4)-(7), which determine the dynamical evolution of the universe at hand, is that (6), (7) and a combination of (4) and (5), can be derived from the Lagrangian:

$$\mathbf{L}_0 = \kappa a^3 (\rho - P) = 3a\dot{a}^2 - 3ak + \kappa a^3 \left[ \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\dot{\sigma}^2 + V(\phi, \sigma) \right], \quad (8)$$

or from the Hamiltonian:

$$\mathbf{H}_0 = 3a^3 \left[ H^2 - \frac{k}{a^2} + \frac{\kappa}{6} (\dot{\phi}^2 - \dot{\sigma}^2) - \frac{\kappa}{3} V(\phi, \sigma) \right]. \quad (9)$$

The aforementioned simple quintom cosmological model comprises of three degrees of freedom, one of which carries negative kinetic energy. Therefore, we can construct a toy-model that consists of three coupled oscillators, one with negative-definite and two with positive-definite kinetic energy. This toy-model, although not exact (i.e re-producing the true system), it presents the same qualitative behavior [23] and thus it can mimic the real

system described by (8) or (9), in the case of  $k = 0$ . In the simple case of a quadratic potential  $V = m^2(\phi^2 + \sigma^2)/2$ , such a model can be formulated using the Lagrangian:

$$\mathbf{L} = \frac{\dot{x}^2}{2} - \frac{\dot{y}^2}{2} + \frac{\dot{z}^2}{2} - \frac{m_x^2 x^2}{2} - \frac{m_y y^2}{2} - \frac{m_z z^2}{2} - \frac{\mu^2 x^2 y^2}{2} - \frac{\lambda^2 x^2 z^2}{2}, \quad (10)$$

or the associated Hamiltonian:

$$\mathbf{H} = \frac{\dot{x}^2}{2} - \frac{\dot{y}^2}{2} + \frac{\dot{z}^2}{2} + \frac{m_x^2 x^2}{2} + \frac{m_y y^2}{2} + \frac{m_z z^2}{2} + \frac{\mu^2 x^2 y^2}{2} + \frac{\lambda^2 x^2 z^2}{2}. \quad (11)$$

As far as the total energy of the system remains constant, the energy of the phantom oscillator could decrease arbitrarily, while the energy of the other two oscillators could increase infinitely. Hence, there is not a stable ground state for the system [23, 21].

We can now derive the Euler-Lagrange equations from (10):

$$\ddot{x} + (m_x^2 + \mu^2 y^2 + \lambda^2 z^2)x = 0 \quad (12)$$

$$\ddot{y} - (m_y^2 + \mu^2 x^2)y = 0 \quad (13)$$

$$\ddot{z} + (m_z^2 + \lambda^2 x^2)z = 0. \quad (14)$$

In the following we assume that  $m_x = m_y = m_z = 0$  and  $\mu^2 = 1$  and  $\lambda^2 = 1$ , so that equations (12)-(14) are reduced to:

$$\ddot{x} = -(y^2 + z^2)x \quad (15)$$

$$\ddot{y} = x^2 y \quad (16)$$

$$\ddot{z} = -x^2 z. \quad (17)$$

The toy-model governed by the equations of motion (15)-(17) mimics the behavior of that of (4)-(7), or in other words the massless quintom cosmological paradigm is qualitatively isomorphic to the constructed system of coupled oscillators. This property allows us to reveal the characteristics of the cosmological scenario by investigating the dynamical evolution of this oscillator system.

## 2.2 Conformally-coupled scalar field cosmology

There are many arguments supporting that in a curved space a scalar field couples non-minimally to the Ricci curvature. The explicit non-minimal coupling to the curvature introduces extra terms in the equations for the scalar fields and the scale factor, allowing for an accelerating expansion for the universe or even for a super-accelerating ( $\dot{H} > 0$ ) universe. Apart from the possibility of explaining the observed recent universe acceleration, there are additional reasons to consider such a model. Non-minimal coupling is introduced by quantum corrections to the action of a scalar field [24], it is necessary for the renormalization of the scalar field theory [25], and it is even required at the classical level to preserve the Einstein equivalence principle or to avoid causal pathologies [26]. Thus, in this work we consider conformally-coupled massive scalar fields.

We use an action including scalar fields with positive, non-minimally-coupled kinetic energy with the Ricci curvature [23]:

$$S = \int d^4x \sqrt{-g} \left[ \frac{R}{2\kappa} - \frac{\xi}{2} (\phi^2 + \sigma^2) R - \frac{1}{2} g^{ab} \nabla_a \phi \nabla_b \phi - \frac{1}{2} g^{ab} \nabla_a \sigma \nabla_b \sigma - V(\phi, \sigma) \right], \quad (18)$$

where  $\xi$  is a dimensionless coupling constant. We are interested in studying the specific case where  $\xi = 1/6$ , since such a choice is an infrared fixed point of the relevant renormalization group [27]. Assuming a potential form  $V = m^2(\phi^2 + \sigma^2)/2$ , we derive the equations of motion [28, 29, 30, 31, 32, 33]:

$$\dot{H} + 2H^2 + \frac{k}{a^2} - \frac{\kappa m^2}{6} (\phi^2 + \sigma^2) = 0 \quad (19)$$

$$\frac{\kappa}{2}(\dot{\phi}^2 - \dot{\sigma}^2) + \kappa H(\phi\dot{\phi} - \sigma\dot{\sigma}) - 3H^2 \left[ 1 - \frac{\kappa}{6}(\phi^2 + \sigma^2) \right] + \frac{\kappa m^2}{2}(\phi^2 + \sigma^2) = 0 \quad (20)$$

$$\ddot{\phi} + 3H\dot{\phi} + \frac{R}{6}\phi + m^2\phi = 0 \quad (21)$$

$$\ddot{\sigma} + 3H\dot{\sigma} - \frac{R}{6}\sigma - m^2\sigma = 0, \quad (22)$$

along with the Hamiltonian constraint as the first fundamental FRW equation:

$$H^2 + \frac{k}{a^2} = \frac{\kappa}{3}\rho. \quad (23)$$

The effective energy density and pressure of the scalar fields, which guarantee energy conservation, are given by [34]:

$$\rho = \frac{1}{2}(\dot{\phi}^2 - \dot{\sigma}^2) + \frac{m^2}{2}(\phi^2 + \sigma^2) + \frac{k}{2a^2}(\phi^2 + \sigma^2) + \frac{1}{2}H\phi(H\phi + 2\dot{\phi}) + \frac{1}{2}H\sigma(H\sigma - 2\dot{\sigma}) \quad (24)$$

and

$$P = \frac{1}{2}(\dot{\phi}^2 - \dot{\sigma}^2) - \frac{m^2}{2}(\phi^2 + \sigma^2) - \frac{k}{6a^2}(\phi^2 + \sigma^2) - \frac{1}{6}[4H(\phi\dot{\phi} - \sigma\dot{\sigma}) + 2(\dot{\phi}^2 - \dot{\sigma}^2) + 2\phi\ddot{\phi} - 2\sigma\ddot{\sigma} + (2\dot{H} + 3H^2)(\phi^2 + \sigma^2)]. \quad (25)$$

Equations (19)-(22) and expression (24) for  $\rho$  are evidently complicated. However, we can reduce the problem to a system of three coupled oscillators with sharply defined energies in a fictitious Minkowski space, repeating the same steps as in the previous subsection. Indeed, inserting the auxiliary variables:

$$x \equiv ma, \quad y \equiv \sqrt{\frac{\kappa m^2}{6}} a\phi, \quad z \equiv \sqrt{\frac{\kappa m^2}{6}} a\sigma, \quad (26)$$

introducing the rescaled (conformal) time  $\eta$  given as  $dt = a d\eta$ , and assuming a quadratic potential  $V = m^2(\phi^2 + \sigma^2)/2$ , the equations of motion are transformed to [35, 36, 37]:

$$x'' = (y^2 + z^2 - k)x \quad (27)$$

$$y'' = -x^2 y \quad (28)$$

$$z'' = x^2 z, \quad (29)$$

where primes denote differentiation with respect to  $\eta$ . Note that these equations can be obtained from the Lagrangian:

$$\mathbf{L}_1 = -\frac{1}{2}(x')^2 + \frac{1}{2}(y')^2 - \frac{1}{2}(z')^2 - \frac{1}{2}x^2 y^2 - \frac{1}{2}x^2 z^2 + \frac{1}{2}kx^2, \quad (30)$$

or from the Hamiltonian:

$$\mathbf{H}_1 = -\frac{1}{2}(x')^2 + \frac{1}{2}(y')^2 - \frac{1}{2}(z')^2 + \frac{1}{2}x^2 y^2 + \frac{1}{2}x^2 z^2 - \frac{1}{2}kx^2. \quad (31)$$

Clearly, the system of equations (27)-(29) is isomorphic to that of (19)-(22). We mention here that  $y$  and  $z$  are a mixture of the gravitational and scalar-field degrees of freedom, whereas  $x$  is associated solely with gravity. Furthermore, as it is required by the first relation of (26), the restriction  $x > 0$  must be applied.

### 3 Dynamical behavior of massless-quintom oscillator-model

In the previous section we formulated the use of oscillator-models in investigating the dynamical behavior of cosmological paradigms. For a spatially flat universe under the massless phantom scenario, the associated system of two coupled oscillators was studied by Castagnino *et al* [37] (see also [23]). In this section we examine the evolution characteristics of the oscillator-model corresponding to the massless quintom, which consists of three degrees of freedom and was formulated in subsection 2.1. In this case the phase-space stability analysis leads to the following results:

-Firstly, we extract the fixed points of the system (15)-(17), defined as those points where the velocities of the oscillators are zero. It is easy to see that these are simply the loci  $(x_0, 0, 0)$  and  $(0, y_0, z_0)$ . In order to examine the stability of these fixed points, as usual we calculate the partial derivatives of the right hand sides of the system (15)-(17) at these points, and we extract the eigenvalues of the corresponding matrix [38]. Since at least one of the eigenvalues is always positive, we conclude that the fixed points are all unstable. In fact, a three-oscillator system with one of them having negative-definite kinetic energy, does not possess positions of equilibrium.

-Due to these instabilities, apart from the fixed points, all the orbits in the phase-space go to infinity as  $t \rightarrow \infty$ . In particular,  $x(t) \rightarrow \infty$  monotonically while  $y(t)$  and  $z(t) \rightarrow 0$  oscillating, a behavior which is independent of the choice of initial conditions. This can be easily verified by simple numerical investigations.

-Cycles (periodic orbits) are not possible. However, chaotic dynamics may appear. This is a robust result for the case of the quadratic (or equivalently the Yang-Mills) potential, and arises from the corresponding extensive studies of the literature (see for example [39]).

-In general, as it has been shown in [23, 37], and taking into account the invariance under the transformation  $(x, y, z) \rightarrow (-x, y, z)$  and  $(x, y, z) \rightarrow (x, -y, z)$  and  $(x, y, z) \rightarrow (x, y, -z)$ , we can consider an asymptotic solution as:

$$y(t) \approx z(t) \approx \frac{\sqrt{2}}{t} \sin\left(\frac{t^3}{3}\right) \quad (32)$$

$$x(t) \approx t^2. \quad (33)$$

In this case the kinetic energy of the  $y$  and  $z$ -oscillators (for large times) is

$$K^{(y,z)} = \frac{(\dot{y}^2)}{2} \approx t^2 \cos^2\left(\frac{t^3}{3}\right). \quad (34)$$

We can see that this expression oscillates with divergent amplitude, while the kinetic energy of the  $x$ -oscillator,  $K^{(x)} = -\frac{(\dot{x}^2)}{2} \approx -2t^2 \rightarrow -\infty$ . This behavior corresponds to the instability described in [21].

Having examined the system characteristics using the auxiliary degrees of freedom (26) we can now transform back to the physical variables  $a$ ,  $\phi$  and  $\sigma$ , under the restriction  $x > 0$ , noting the necessary inversion between variables indicated in [23]. Doing so our results can be re-written as follows:

-The fixed points of the cosmological model are just  $(a_0, 0, 0)$  (we discard the family  $(0, y_0, z_0)$  since it corresponds to the non-physical case of a universe with zero scale factor). Thus, they correspond to Minkowski spaces with constant scale factor, i.e without expansion. The fact that they are unstable, i.e the absence of attractors, implies that an arbitrary small perturbation can lead to the aforementioned diverging orbits.

-In particular, the previously analyzed behavior of  $x(t)$ ,  $y(t)$  and  $z(t)$  implies that the scale factor  $a$  goes to infinity, while the scalar fields are oscillating, in general with a varying period. Thus, we conclude that we always acquire an expanding universe. This is true even in the case where the system lies initially in one of the fixed points (constant scale

factor and zero derivative), since an arbitrary small perturbation is sufficient to lead it to the aforementioned expanding case. Finally, as was shown in [23], the asymptotic solutions (32),(33) correspond to a matter-dominated universe with  $a(t) = a_0 t^{2/3}$ . That is, in this model there is no mechanism that can end the expansion, either by reversing it to contraction or by stabilizing the universe to a steady-state type. Thus, increasing dilution and the “thermal death” of the universe are inevitable.

-Periodic orbits do not exist. That is, a massless quintom cannot drive cyclic universes [40]. However, since chaotic behavior is possible, we conclude that we can obtain chaotic cosmological evolution. Indeed, FRW cosmologies are known to present chaotic behavior [41].

## 4 Dynamical behavior of conformally-coupled quintom oscillator-model

Let us now examine the dynamical characteristics of a massive conformally-coupled quintom, extending the associated oscillator-model formulated in subsection 2.2 (see [42] for the corresponding problem for a phantom field). The Klein-Gordon equations of the scalar fields are:

$$\ddot{\phi} + 3H\dot{\phi} - m^2\phi - \xi R\phi = 0 \quad (35)$$

$$\ddot{\sigma} + 3H\dot{\sigma} + m^2\sigma + \xi R\sigma = 0. \quad (36)$$

In terms of the auxiliary variables  $x$ ,  $y$  and  $z$  defined in (26) and the conformal time  $dt = ad\eta$ , these field equations can be derived from the Lagrangian:

$$\mathbf{L}_2 = \frac{x'^2}{2} + \frac{y'^2}{2} + \frac{z'^2}{2} + \frac{x^2 y^2}{2} + \frac{x^2 z^2}{2} - \frac{1}{2} k x^2, \quad (37)$$

or from the Hamiltonian:

$$\mathbf{H}_2 = \frac{x'^2}{2} + \frac{y'^2}{2} + \frac{z'^2}{2} - \frac{x^2 y^2}{2} - \frac{x^2 z^2}{2} + \frac{1}{2} k x^2. \quad (38)$$

Note that the Lagrangian (37) is equivalent to  $\mathbf{L}_1$  of equation (30) (apart from an overall sign) provided that the conformally-coupled scalar is turned into a phantom field. The dynamical system in this case is:

$$x'' = (y^2 + z^2 - k)x \quad (39)$$

$$y'' = x^2 y \quad (40)$$

$$z'' = x^2 z. \quad (41)$$

In order to perform a stability analysis, we have to distinguish between the various  $k$ -cases.

-Firstly, in the case of a flat universe ( $k = 0$ ) we can see that the fixed points are just  $(x_0, 0, 0)$ , while for  $k = 1$  they are  $(x_0, \pm\nu, \pm\sqrt{1-\nu^2})$ . Both these loci correspond to empty Minkowski spaces. However, for the case of a closed universe ( $k = -1$ ) there are not fixed points, apart from the trivial case of a zero-scale-factor universe.

-It is straightforward to see that these fixed points are unstable, and that all the (non-stationary) orbits in the phase-space go to infinity as  $t \rightarrow \infty$  [37]. In particular, numerical investigation shows that for  $k = 0, 1$ ,  $x(t) \rightarrow \infty$  monotonically while  $y(t)$  and  $z(t) \rightarrow 0$  oscillating. In fact, the oscillatory nature of the solutions for  $k = 0$  is a general feature of non-minimally coupled scalar fields with  $\xi > 0$ , even in the case where  $\xi \neq 1/6$  [30]. Finally, in the case  $k = -1$  there is not a qualitatively general asymptotic behavior and the system evolution can be more complicated. In this case, as is confirmed by numerical integration, the scale factor can sometimes decrease.

-Thus, in the cases of open and flat universes, all the solutions that are not stationary represent universes expanding to infinity. However, due to the instabilities, we conclude that

even an arbitrary small perturbation can bring the universe out of stationarity, and lead it to an everlasting expansion. Therefore, independently of the initial conditions, we always obtain an expanding universe. The expansion cannot be reversed or end, and a complete dilution is inevitable. Finally, note that contrary to the case of the previous section, there is not a specific quantitatively asymptotic behavior. Thus, the Hubble parameter can be constant, or increasing, leading to an accelerating or even super-accelerating universe. On the other hand, for the case of a closed universe, the system evolution is more complex and difficult to be outlined.

-Closed orbits in phase-space do not exist, that is a massive conformally-coupled quintom cannot bring about a cyclic universe [40]. However, due to the quadratic nature of the potential, chaotic behavior is possible, which, as we have already mentioned, is expected in FRW cosmologies [41]. Finally, chaoticity is more easily obtained in the case of a closed universe ( $k = -1$ ), where the dynamics of the system can be more complicated.

## 5 Conclusions

In this work we investigate the evolution characteristics of quintom universes, using oscillator-models as an isomorphic description. Indeed, one can construct such models which present all the relevant information, examine their dynamical behavior under stability analysis, and finally transform the results back to the cosmological picture. In particular, in [37] the authors have constructed a toy-model with two degrees of freedom in order to isomorphically describe the cosmological paradigm of a minimally-coupled massless phantom field and a massless graviton in physical time. In such a case the perturbations of the model are unstable and this feature led the authors of [21] to extend it by the insertion of a negative mass term to the potential, necessary for the stabilization of perturbations (the model of Castagnino *et al.* of [37] could be recovered by forcing the phantom to be massless).

The dynamical system (15)-(17) of the present work constitutes a toy-model for a minimally coupled massless quintom field and a massless graviton in physical time, since in this case the isomorphic description requires three degrees of freedom. Expressing the results in the physical framework we find that the fixed points of the system are just the Minkowski spaces, but are unstable under, even arbitrarily small, perturbations. Moreover, there are no attractor points and all the phase-space orbits go to infinity with increasing time. In particular, in this asymptotic case we obtain a matter-dominated universe with  $a(t) = a_0 t^{2/3}$ . Thus, we conclude that, independently of the initial conditions, the universe is always led to an eternal expansion.

The dynamical system (27)-(29) describes two conformally-coupled massive scalar fields. For flat and open geometries, all the fixed points are empty Minkowski spaces and they are unstable. With increasing time the scalar fields go to zero in an oscillatory way. The obtained evolution reveals that the matter content dilutes progressively up to its complete evanescence while the universe expands. Such a behavior is consistent with the phenomenology of non-minimally-coupled scalar fields. In addition, periodic orbits are not present, and thus this cosmological paradigm cannot drive cyclic universes. However, chaotic dynamics may arise, a feature already known for FRW cosmologies. On the other hand, for a closed universe, the dynamics of the system is more complex, and it is hard to determine even a general qualitative behavior.

The dynamical system (39)-(41) corresponds to a conformally-coupled massive quintom field. Investigating the behavior of its phase-space we see that for a flat or open universe the fixed points represent Minkowski spaces, which are unstable under perturbations. Thus the universe is expanding to infinity. However, this scenario allows for a general (unspecified) Hubble parameter, i.e for either accelerating or even super-accelerating universe. Closed orbits, corresponding to cyclic behavior, do not appear but chaoticity does. Finally, for a closed universe the system evolution can be more complicated.

In conclusion, we observe that we can extract qualitative as well as (asymptotically)



quantitative characteristics of various quintom paradigms, by investigating the corresponding isomorphic coupled-oscillator models. The stability analysis and the obtained asymptotic behaviors show that the quintom scenario is consistent with cosmological observations.

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